On the Quantum Cat and Sawtooth Maps-Return to Generic Behaviour

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Abstract

The quantization of the continous cat maps on the torus has led to rather pathological quantum objects [6]. The non-generic behaviour of this model has led some to conclude that the Correspondence Principle fails in this case [2]. In this note we introduce the quantum sawtooth models, since this is the natural family to which the cat maps belong. Thus, a simple propagator depending on a parameter is constructed which for integer values of the parameter becomes pathological quantum cat maps, while away from integer values we find a return to the generic behaviour of non-integrable quantum systems.

1 Introduction

In this note we will quantize the sawtooth maps [1]. These are generalizations of the cat maps [6]. The quantum cat maps have many non-generic features, resulting from the periodicity of the propagator. There have been speculations about the "failure of the correspondence principle", using the quantum cat map and arguments about their algorithmic complexity (or lack of it) [2]. The arguments of Joseph Ford and his coworkers relies on the fact that the quantum dynamics of the cat map is identical to the classical motion on a rational grid. The periodicity of every rational grid in the case of cat maps is a peculiarity, and results in the periodicity of the quantum propagator. All the eigenvalues of the quantum cat map are roots of unity.

The eigenvalues are highly degenerate, and the work of J.P. Keating [3] has shown that in the classical limit the spectrum becomes infinitely degenerate, although very slowly.

Given the controversy generated by the quantum cat maps, it is natural to ask what happens in the case of their most natural generalizations, the quantum sawtooth maps. Surprisingly, to the best of our knowledge, this has not been done, although the quantization itself is straightforward. The classical mechanics of the sawtooth maps has been investigated by several authors [1,4], I.C. Percival in [5], with a view towards studying transport in the presence of partial barriers such as cantori. I.C. Percival and his coworkers have developed symbolic dynamics for these maps [1], and N. Bird and F. Vivaldi [4] have found their periodic orbits. *We find that the quantized sawtooth maps are not periodic, their eigenangles are all irrational multiples of* 2π*, and they display level repulsion characteristic of quantum chaotic systems.* We will scale the eigenangles by 2π, so that when we talk of "rational eigenangles", it means that the eiganangles are rational fractions of 2π.

We notice that there is a log time, a time when quantum interference effects dominate and destroy the picture of wavepackets propagating as classical Liouville phase space densities. When K, the real parameter in the family, is very close to an integer, the operator is nearly periodic initially, but with increasing time, the periodicity is lost. Related to this is the fact that the slightest perturbation of the cat map, (making K not an integer), seems to produce irrational eigenangles (when the sawtooth map is hyperbolic). These form bands that are clustered around the eigenangles of the nearby quantum cat map. The band structure disappears rapidly when we move away from the cat map. The differences between the quantum cat maps and sawtooth maps is allied to the mathematical problem of the differences between the complete and incomplete Gauss sums of number theory.

2 The Classical Map

We will very briefly describe the classical map. Consider a free particle that is subjected to time periodic impulses due to a force $F(q)$, given by:

$$F(q) \;=\; K\,\mathrm{Saw}(q), \tag{1}$$

where

$$\begin{array}{rcl} \mathrm{Saw}(q) & = & q \quad (-1/2 \leq q < 1/2), \\ \mathrm{Saw}(q) & = & \mathrm{Saw}(q+1). \end{array} \tag{2}$$

This gives the impulse the shape of a sawtooth, and the map its name [1]. The Hamiltonian for this system can be taken to be

$$H(q,p,t) = \frac{p^2}{2} - \frac{K\,(\mathrm{Saw}(q))^2}{2} \sum_{n=-\infty}^{\infty} \delta(t-n). \tag{3}$$

The potential is periodic with period 1. The Hamiltonian equations of motion give us the map

$$\begin{array}{rcl} q' & = & q + p' \\ p' & = & p + K\,\mathrm{Saw}(q). \end{array} \tag{4}$$

When K is not an integer this map has a discontinuity at half integer points. The sawtooth on the torus is obtained by imposing periodic boundary conditions in both q and p. This means that we take the above map *mod* 1. We have followed I.C. Percival [1] and taken the phase space to be the "chosen torus", centered at the origin, rather than the usual torus. Then there is only one discontinuity at the point $q = 1/2$. When K is an integer this discontinuity vanishes, as it gets "dissolved" by the modulo operation; these are cat maps.

The important point we note is that the potential is already periodicised. Cat maps can also be obtained from the *non periodic* potential $-Kq^2/2$, but the sawtooth maps *cannot*. To see this imagine that the infinite phase plane is tessellated by fundamental squares. Then for integer values of K the linear map

$$\begin{array}{rcl} q' & = & q + p' \\ p' & = & p + K\,q. \end{array} \tag{5}$$

obtained from the unperiodicised potential $-K\,q^2/2$ takes equivalent points to equivalent points. Two phase points are equivalent if they differ by an integer vector. If we *retain* this potential and proceed with the quantization of the Hamiltonian of eqn.(3), requirements of periodicity will naturally force us to restrict ourselves to cat maps. Indeed this is the procedure of Joseph Ford and his coworkers [2], for although their quantization method is general enough the chosen potential was restrictive.

The sawtooth map on the torus is unstable for $K > 0$ and $K < -4$. The stable regime is a curious map filled with many elliptic islands, this is

illustrated in fig.1. K=0 is the case of a free rotor. All the periodic points of the unstable maps are hyperbolic [1]. Periodic orbits of cat maps can be used to find them [4].

3 The Quantum Sawtooth

3.1 The Propagator

The quantization of the Hamiltonian, eqn. (3), after imposing periodic boundary conditions on position and momentum, give us the quantum sawtooth propagator. The quantization of J. Ford et. al. [2] is itself our starting point. We impose *periodic* boundary conditions on the states. The Planck's constant $\hbar$ is related to the dimensionality, N, of the finite Hilbert space by the relation $2\pi\hbar = N^{-1}$. The periodicised position eigenstates are denoted as $|q_n\rangle$ and the periodicised momentum states are denoted as $|p_m\rangle$, $m,\, n = -N/2, \ldots, N/2-1$. The transformation functions are discretised plane waves,

$$\langle q_n|p_m\rangle \;=\; \frac{1}{\sqrt{N}} e^{2\pi i m n/N}. \tag{6}$$

The position and momentum eigenvalues are $n/N\,; n = -N/2, \ldots, N/2-1$.

The unitary propagator obtained by integrating the Hamiltonian of eqn.(3), quantized canonically, over one time step is

$$\hat{U} \;=\; \exp\left(-i\hat{p}^2/2\hbar\right)\, \exp\left(iK(\mathrm{Saw}(\hat{q}))^2/2\hbar\right). \tag{7}$$

The first term of the R.H.S. of the above equation is the propagator corresponding to the free rotation, we denote it as $\hat{U}_0$, the second part arises from the "kick" or the impulse and is denoted as $\hat{U}_1$. $\hat{U}$ is still the propagator for the map on the whole plane. The restriction to a torus is achieved quantally by requiring that the action of the unitary operator maintains the periodicity of the discrete toral states (that are Dirac delta combs). To implement this, first consider the action of the free propagator $\hat{U}_0$ on the discrete toral states,

Thus consider,

$$\langle q_n|\hat{U}_0|p_m\rangle \;=\; e^{-i\pi m^2/N}.\, e^{2\pi i m n/N}. \tag{8}$$

Here we have used the relation $2\pi\hbar = N$. The requirement that the above be periodic in both m and n with a period N, implies that N be even. We

will henceforth assume that this is the case. If we had chosen anti-periodic boundary conditions on the states we would have required N to be odd. Similarly the mixed representation of the kick operator is

$$\langle p_m|\hat{U}_1|q_n\rangle \;=\; e^{i\pi NK(\mathrm{Saw}(n/N))^2}.\, e^{-2\pi i mn/N}. \tag{9}$$

The periodicity in n and m follows immediately from the periodicity of the sawtooth potential. We in particular do *not* require that K be an integer. J. Ford et. al. used identical quantization procedures [2], but as noted earlier, the potential was taken to be (essentially) $-Kq^2/2$, which leads to the factor $e^{i\pi Kn^2/N}$ as the first term in the R.H.S. of the above equantion. Imposing periodicity on this factor would then lead to the restriction of K to the integers. In fact, as we have seen this is true even classically, if we start with the harmonic oscillator potential, instead of the nonlinear periodic sawtooth potential.

Such quantizations can be carried out for any periodic potential. When the potential is a cosine, the map is the famous standard map. The quantum propagator is an $N \times N$ matrix, when restricted to act on the Hilbert space of N states. Then we have $\mathrm{Saw}(n/N) = n/N$. We can now put together the operators $\hat{U}_0$ and $\hat{U}_1$, and write the quantum sawtooth map in the position representation as

$$\langle q_n|U|q_{n'}\rangle \;=\; \frac{1}{N}e^{i\pi Kn'^2/N}.\sum_{k=-N/2}^{N/2-1} e^{2\pi k(n-n')/N}\, e^{-i\pi k^2/N}. \tag{10}$$

The sum above simplifies upon using the Poisson summation formula, and we get the final form of the propagator as

$$\langle q_n|U|q_{n'}\rangle \;=\; \frac{e^{-i\pi/4}}{\sqrt{N}}e^{i\pi Kn'^2/N}\, e^{i\pi(n-n')^2/N}. \tag{11}$$

We have dropped the hats for the operators on the torus, which are simply finite unitary matrices. When K is an integer the above is a quantum cat map, otherwise it is a discontinuos quantum sawtooth map. The propagator is thus a very simple one, and is the natural generalization of the quantum cat maps of Hannay, Berry and Ford [6,2]. The operator U has all the features we have noted in the introduction. For integer K it is a periodic operator with all rational eigenangles, i.e., there is an integer $n(N)$ (n is not to be

confused with the position labelling) such that $U^{n(N)} = e^{i\phi(N)} I_N$. Here I_N is the $N \times N$ identity matrix, and $\phi(N)$ is a real phase. When K is not an integer and the operator is not periodic, all the eigenangles are irrational.

We have to qualify the last statement, since : a) it is a numerical observation, b) there are cases for $-4 < K < 0$, when there are some rational eigenangles. These correspond to stable sawtooth maps, when $\cos^{-1}(K+2/2)$ is a rational multiple of π. If there is an elliptic fixed point at the origin for a linear map on the plane, then the eigenangles form a harmonic oscillator spectrum, with eigenangles given by the equation

$$(l + 1/2)\cos^{-1}(tr/2), \quad l = \ldots, -2, -1, 0, 1, \ldots. \tag{12}$$

where tr is the trace of the classical matrix describing the map (for instance, see ref.7). In the case of the sawtooth map the trace is $K + 2$. If the stable fixed point at the origin has a large elliptic island, see fig.1, which does not "feel" the nonlinearity of the map, then many sequences of eigenangles of the operator U are well predicted by the above equation.

The classical map has the symmetry of reflection about the center of the square ($q \to -q$, $p \to -p$). This symmetry is present in the quantum operator U, as $U_{nn'} = U_{-n-n'}$. The choice of origin at the center of the square makes the symmetry matrix have the form

$$P_N = \begin{pmatrix} 1 & 0 \\ 0 & R_{N-1} \end{pmatrix}, \tag{13}$$

where R_{N-1} is the $N-1 \times N-1$ matrix with 1 along the secondary diagonal, and 0 elsewhere. Thus $(R_{N-1})_{mn} = 1$ if $m + n + 1 = N - 1$ and zero otherwise. Since $P_N^2 = I_N$, and $[P_N, U] = 0$, the eigenstates of the propagator can be separated according to their parity. Any eigenvector is of the form

$$|\psi_\pm\rangle = \begin{pmatrix} \alpha \\ |\psi_1\rangle \\ \pm R_{N/2-1}|\psi_1\rangle \end{pmatrix}, \tag{14}$$

where α is the component $\langle -N/2|\psi_\pm\rangle$. This implies that odd parity eigenstates should have $\alpha = 0$. Thus there are $N/2 - 1$ odd parity states and $N/2 + 1$ even parity states. When finding the nearest neighbour distribution of the eigenangles, we will select only the eigenangles corresponding to odd parity states.

3.2 Numerical Results

The most commonly used statistics is the that of the nearest neighbour spacing. Shown in fig.2 are the nearest neighbour spacing distribution for $K = 2.25, 2.50, 2.55, 3.25$. The level repulsion is apparent. The statistics is for the 149 odd parity states, when $N = 300$. Better statistics would require more eigenangles, but the essential feature of level repulsion is clear enough. The statistics when K is very close to an integer value, when the sawtooth map is almost a cat map, is bound to be rather peculiar. In these cases there is, as has been noted above, a band like structure, so that levels cluster around well separated eigenangles. When we move away from the integer value the band like structure quickly gives way to a more uniformly spread distribution exhibiting level repulsion. This is the case of the fig.3, where we have the nearest neighbour spacing distribution for the $K = .01, .1, 3.01, 3.1$.

Classically there is a significant difference between the case when K is close to zero, and when K is close to some other integer. In the former case we have just moved away from the integrable free rotor at $K = 0$. The KAM theorem conditions are not met, as the sawtooth map is not smooth, hence we have no deformed tori. The map becomes immediately globally chaotic for a positive K value. However for small positive K there are significant partial barriers to transport, cantori made of parts of the stable and unstable manifolds. These cantori are less important as barriers when the K value is large. The Poisson distribution of the eigenangles for $K = 0.01$ and the level repulsion for $K = 3.01$, shown in fig.3 is a quantum manifestation of this difference. Also compare the cases, $K = .1$ and $K = 3.1$. That the eigenvalue statistics can be affected by classical transport properties has been exhibited before [8]. The sawtooth maps provide another example for this phenomenon, which needs more study.

Figs.4,5,6 show contours of the autocorrelation functions. We use the coherent states developed by M. Saraceno [9] adapting it to periodic boundary conditions. It is a phase space representation that allows the classical structures of quantized toral maps to be more easily identified. If $|p, q\rangle$ is such a coherent state, p and q take values on the classical $N \times N$ rational grid of the torus. It is a state that is highly concentrated at (p,q), in the sense of a minimum uncertainty wavepacket. Thus the autocorrelation $|\langle p, q|U^t|p, q\rangle|^2$ is the probablility that a wavepacket initially centered at (p, q) "comes back" to (p, q) after t time steps. For quantum cat maps the autocorrelation is

periodic in time, due to the periodicity of the propagator itself. However sawtooth maps that are far from cat maps, show autocorrelations like other quantum systems, such as the quantum baker's map [10]. Fig.4 shows autocorrelations for the cat $K = 2$. fig.5 is the case of the sawtooth $K = 2.25$. fig.6 is the case of the sawtooth $K = .5$. The value of the inverse of Planck's constant in all the figures is $N = 48$.

In all the figures the map has been shifted to the usual torus $[0, 1) \times [0, 1)$. This does not affect the quantum or classical system in any essential way. The fixed point at the origin now gets shifted to the point $(1/2, 0)$. The quantum cat map of fig.4 behaves as expected, it is periodic. This results in the "emptiness" of fig.4, n=8, when the propagator fixes all wavepackets. The periodicity of the propagator is the *classical* periodicity of the $N \times N$ rational grid, consisting of partitions of equal area [6,2]. F.J. Dyson and H. Falk [12] have given bounds of this periodicity for the case of Arnold's cat map, corresponding to the case of $K = 1$. If m_N is the period of the lattice (and of the quantum propagator) a lower bound has been established as $m_N > [\log N/ \log \lambda]$, (for more complete statement of bounds see ref.12). This surprisingly coincides with the so called log time [11] when classical-quantum correspondence breaks down due to interference.

Thus if we assume that such lower bounds are valid in other cat maps (this is not so hard to believe as these bounds are derived from the divisibilty properties of Fibonnacci numbers) we see that the periodicity of the propagator must be a post log time phenomenon. This is to be expected as the reconstruction of the wavefunctions under a quantum cat map proceeds due to some kind of coherent interference. In fig.4 the first two time steps show significant peaks at classical periodic orbits and nowhere else. The log time for this case of $N = 48, K = 2$ is 2. Hence there are strong interference effects after this time, yet the structures produced are very regular, resulting finally in the identity operator at $n = 8$, apart from a phase factor.

The post log time structures of the quantum cat map are thus very special. There are too many classical periodic orbits such that if each is assinged a phase space volume h $(= 1/N)$ there would be too many to fit the unit square. Yet there are coherent structures around the periodic orbits. For instance in fig.4, n= 4, the uniform striations are precicely the lines along which the classical orbits of period 4 lie. It is said that there "is no $\log(1/\hbar)$ problem for the cat maps" [3]. If we view the log time as heralding the onset of quantum interference effects, we can only surmise that such effects are

very coherent and special for the cat map.

For the sawtooth maps the log time surfaces in a generic manner. The larger eigenvalue (inverse of the smaller one, as map preserves area) of the classical map is

$$\lambda = \frac{(2+K) + \sqrt{K(K+4)}}{2}. \tag{15}$$

The Lyapunov exponent, Λ is $\log \lambda$. Thus when $K = 2.25$ and .5 the exponents are $\Lambda_{2.25} = \log 4$ and $\Lambda_{.5} = \log 2$. The log time is then $\log(48/2)/\Lambda$. The log time for $K = .5 \approx 5$ is twice that of the case $K = 2.25$. Thus we see that after the second time step (n=2), in fig.5 interference effects set in and the peaks that are at classical periodic points are no longer clearly visible. While the corresponding situation for $K = .5$ in fig.6 shows the longer log time, and after about 5 time steps the interference effects are visible. When K is very close to an integer the correlations are once more very close to that of the nearby cat map; with increasing time, however, interference effects destroy the periodicity. Figs. 5 and 6 also show some coherent reconstructions after the log time, but more study is needed to understand these.

4 Conclusions

In this note we have begun the study of quantum sawtooth maps. They place the non-generic quantum cat maps in a family with generic behaviour. We have found level repulsion and the existence of a log time in the sawtooth maps. The log times agree well with the expectations. The propagator becomes periodic when the sawtooth maps become cat maps, otherwise they have no exact periodicity. Thus we cannot expect that the periodic orbit sums for the sawtooth maps will be exact. Unlike the cat maps such sums will once more be semiclassical approximations.

The singular character of the quantum cat maps is reflected in the degeneracy of the eigenangles. While quantized chaotic systems in general do not have any degeneracies, the cat maps become increasingly degenerate as we approach the classical limit. The sawtooth maps have no such degeneracies and the classical limit of this map may be expected to behave in the usual manner.

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